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On the equivalence of the integral and the differential exact solution generation methods for the one-dimensional Schrödinger equation

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Abstract. It is demonstrated that the integral exact solution generation methods for the one-dimensional Schrödinger equation based on the Gelfand–Levitan formalism are in some cases equivalent to the differential ones based on the n -order Darboux transformation. Some new exact solvable potentials are generated from the effective Coulomb potential and the harmonic oscillator potential. A new form of n -soliton potential (i.e. reflectionless potential with n discrete energy levels disposed in a desirable manner) based on an explicit expression for an n -order Wronski determinant constructed from hyperbolic functions and its orthonormal discrete spectrum eigenfunctions are given.

1. Introduction

Two different ways are known to construct the new one-dimensional Schrödinger equation exactly solvable potentials. The first, derived from the inverse scattering problem, is based on the Gelfand–Levitan or Marchenko equation [1–3] (see also for an excellent survey, [4]) and represents an integral transformation of the solutions of the initial Schrödinger equation. The second uses the differential transformation proposed by Darboux [5], developed by Crum [6] and Krein [7], and applied in a restricted form to the supersymmetric quantum mechanics (there are many papers on this subject; see, for example, [8]). The two methods lead to similar results, but the origin of this similarity has not been properly discussed. In a recent paper [9] there are some discussions and the connection of the degenerate kernel integral transformation with the Darboux transformations is established. We obtained a similar result but in a different way. In this paper, with the help of some properties of an n -order Wronski determinant, it is demonstrated that the differential n -order Darboux transformation presented in its general form by Crum [6] may have the form of an integral transformation. The integral exact solution generation methods based on this transformation are consequently equivalent to the differential ones based on the n -order Darboux transformation.

It is worthwhile mentioning an interesting work by Mielnik [10] concerning the use of the factorization method to construct new potentials with an energy spectrum coinciding with that of the known Hamiltonian. One can find the discussion about the connection of the factorization method with the n -order Darboux transformation in [11].

One usually constructs the Darboux transformation operator with the help of the ground state function or with a function which does not belong to the discrete spectrum of the initial Schrödinger equation [12–14]. This limitation is due to the condition that the

potential difference between a new potential and an initial one is a regular function. In this way, by performing a chain of Darboux transformations we obtain a chain of well defined Hamiltonians. Some properties of the Hamiltonians thus obtained are discussed in [15–17]. In this connection we will point out that there are other ways to construct well defined potentials using the Darboux transformation. The first is to use two-by-two juxtaposed discrete spectrum wavefunctions as transformation functions in the chain of transformations. This possibility follows from Krein's theorem [7] and was recently rediscovered by Adler [18]. The chain of the Hamiltonians will in this case contain ill-defined elements which, nevertheless, do not affect the regularity of the final potential. From the point of view of supersymmetric quantum mechanics we shall obtain here higher-derivative supercharges, forming together with the super-Hamiltonian a high-order superalgebra [11]. The second is to use the integral representation of the Darboux transformation. In this paper, the direct correspondence between the differential n -order Darboux transformation and an integral one is established.

We will note that this representation can be helpful in the construction of soliton-like solutions of the nonlinear differential equation. The Darboux transformation seems very fruitful in soliton theory (see [19] and references therein). In this connection a new representation of the well known n -soliton potential with n discrete spectrum energy levels disposed in the desired manner based on a new closed form for the n -order Wronski determinant constructed from the hyperbolic functions is given. The orthonormal discrete spectrum wavefunctions of this potential are cited.

As an example, the effective Coulomb potential is studied. We give two new regular potentials with a hydrogen-like spectrum. These potentials are unlike the effective Coulomb one around $0.1 < x < 10$ au and tend to zero as z/x when x tends to infinity. Another example is two potentials with a quasi-equidistant spectrum generated from the harmonic oscillator potential. One potential proceeds from the quadruple Darboux transformation and the other is a double well, and the energetic level of the ground state touches the potential curve at its maximum.

2. Generalized Darboux transformation

Let us consider the one-dimensional Schrödinger equation (setting $\hbar = 2m = 1$)

$$H_0\psi_E(x) = E\psi_E(x) \quad H_0 = -d^2/dx^2 + V(x) \quad x \in [a, b], \quad (1)$$

(We do not exclude infinite intervals for variable x .) It is supposed that we know the general solution of this equation for all values of parameter E (complex in general). Potential $V(x)$ is supposed to be a sufficiently smooth function in $[a, b]$. In this paper we shall consider only those solutions of equation (1) which satisfy the zero boundary condition and the condition $|\psi'_E(x)| < \infty$ at least in one of the bounds of interval $[a, b]$. If in addition we impose (in the case of a potential well) the same boundary condition in another bound of interval $[a, b]$, we obtain the discrete spectrum of $E = E_n$ (if it exists).

Single Darboux transformation permits one to obtain the general solution of another Schrödinger equation

$$H_1\psi_E(x) = E\psi_E(x) \quad H_1 = -d^2/dx^2 + U(x) \quad x \in [a, b] \quad (2)$$

acting by differential first-order linear operator L on the general solution of equation (1)

$$\varphi_E(x) = L\psi_E(x) \quad L = L_0(x) - d/dx. \quad (3)$$

The potential difference $A(x) = U(x) - V(x)$ and function $L_0(x)$ from formula (3) must be determined on the basis of some solution $\psi_\alpha(x)$ of equation (1) with $E = \alpha$ (α being an arbitrary parameter) called the transformation function

$$L_0(x) = \psi'_\alpha(x)/\psi_\alpha(x) \quad A(x) = -2L'_0(x). \tag{4}$$

This procedure being applied n times leads to the generalized Darboux formula obtained by Crum [6]

$$\varphi_E(x) = \frac{W_{n+1}(x)}{W_n(x)} \quad A(x) = -2 \frac{d^2}{dx^2} \log W_n(x). \tag{5}$$

Function $W_n(x)$ coincides with Wronski determinant $W_n(x) = W(\psi_{\alpha_1}, \dots, \psi_{\alpha_n})$ (W being the usual symbol for a Wronskian) if potential $V(x)$ and its $n - 3$ derivatives are continuous functions, $W_{n+1}(x)$ is the Wronskian of the functions $\psi_E(x), \psi_{\alpha_1}(x), \dots, \psi_{\alpha_n}(x)$. Every derivative from these Wronskians whose order exceeds the first can be expressed from equation (1) either by the function itself or by its first derivative. The derivative of even order $\psi_\alpha^{(2k)}(x)$ must be substituted by $(-\alpha)^k \psi_\alpha(x)$ and that of odd order by $(-\alpha)^k \psi'_\alpha(x)$. The Wronski determinant after these substitutions converts into the determinant called by us the Krein determinant and denoted as $W_*(\psi_{\alpha_1}, \dots, \psi_{\alpha_n})$.

The double Darboux transformation leads to the known integral transformations [3, 4]. For the first transformation, we use some function $\psi_\alpha(x)$ and for the second $\psi_\alpha^{-1}(x)(C_1 + C_2 \int^x \psi_\alpha^2(z) dz)$, which is the general solution of equation (2) with $E = \alpha$. We shall now show that the same transformation comes directly from expression (5) at $n = 2$ and generalizes this result to arbitrary n .

If instead of the Wronskian $W_2(x)$ we introduce the function

$$W_{\alpha,\beta}(x) = \frac{W_2(x)}{\alpha - \beta} = (\alpha - \beta)^{-1} [\psi_\alpha(x)\psi'_\beta(x) - \psi'_\alpha(x)\psi_\beta(x)] \tag{6}$$

and decompose the determinant $W_3(x)$ in the first column we obtain for the function $\varphi_E(x)$ the following expression:

$$\varphi_E(x) = (E - \alpha) [\psi_E(x) - \psi_\alpha(x)W_{\alpha,\beta}^{-1}(x)W_{\beta,E}(x)]. \tag{7}$$

When using the property $W'_{\alpha,\beta}(x) = \psi_\alpha(x)\psi_\beta(x)$, we can recalculate $W_{\alpha,\beta}(x)$ using the following integral:

$$W_{\alpha,\beta}(x) = \int^x \psi_\alpha(z)\psi_\beta(z) dz + C \tag{8}$$

where the constant C is to be determined by the bottom integration bound. From this expression, neglecting the constant factor $\beta - E$, we obtain the integral representation for $\varphi_E(x)$

$$\varphi_E(x) = \psi_E(x) - \int^x K(x, z)\psi_E(z) dz \tag{9}$$

which coincides with that used in [1-4] except for the form of the operator kernel $K(x, z)$. The constant C connected with the function $W_{\beta,E}(x)$ is omitted in expression (9). This can be achieved by a proper choice of the low integration bound in which this function must vanish.

The operator kernel from expression (9)

$$K(x, z) = \psi_\alpha(x)W_{\alpha,\beta}^{-1}(x)\psi_\beta(z) \tag{10}$$

differs from that used by previous authors [1-4] by the condition $\alpha = \beta$. We can realize the limit $\beta \rightarrow \alpha$ in function $W_{\alpha,\beta}(x)$ in different ways. This is due to the fact that

equation (1) with given $\beta = E$ has two independent solutions ψ_β and $\tilde{\psi}_\beta$ with the property $W(\psi_\beta, \tilde{\psi}_\beta) = \text{constant}$. If we use the combination $\psi_\beta(x) + (\alpha - \beta) \times \text{constant} \times \tilde{\psi}_\beta(x)$ as $\psi_\beta(x)$ in formula (8), then $W_{\alpha,\beta}(x) \xrightarrow{\beta \rightarrow \alpha} W_{\alpha,\alpha}(x)$ which should be calculated by the same formula (8) with an arbitrary value of the constant C . The formulae (8)–(10) perfectly coincide with those reported in [1, 2].

3. General integral transformation

Let us consider the general transformation presented by formulae (5) with even $n = 2m$. The Krein determinant $W_{2m}(x) = W_*(\psi_{\alpha_1}, \dots, \psi_{\alpha_n})$ has the following remarkable property [11]: it is proportional to the determinant of the $m \times m$ matrix composed of the functions $W_{\alpha_i,\alpha_j}(x)$

$$W_{2m}(x) = \prod_{i \neq j}^{m^2} (\alpha_i - \alpha_j) \times \begin{vmatrix} W_{\alpha_1,\alpha_2} & W_{\alpha_1,\alpha_4} & W_{\alpha_1,\alpha_6} & \dots & W_{\alpha_1,\alpha_n} \\ W_{\alpha_3,\alpha_2} & W_{\alpha_3,\alpha_4} & W_{\alpha_3,\alpha_6} & \dots & W_{\alpha_3,\alpha_n} \\ W_{\alpha_5,\alpha_2} & W_{\alpha_5,\alpha_4} & W_{\alpha_5,\alpha_6} & \dots & W_{\alpha_5,\alpha_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_{\alpha_{n-1},\alpha_2} & W_{\alpha_{n-1},\alpha_4} & W_{\alpha_{n-1},\alpha_6} & \dots & W_{\alpha_{n-1},\alpha_n} \end{vmatrix}. \tag{11a}$$

For the determinant $W_{2m+1}(x) = W_*(\psi_E, \psi_{\alpha_1}, \dots, \psi_{\alpha_n})$ we can obtain the following expression:

$$W_{2m+1}(x) = \prod_{i \neq j}^{m^2} (\alpha_i - \alpha_j) \prod_{k=1}^m (\alpha_{2k} - E) \times \begin{vmatrix} \psi_E & W_{E,\alpha_2} & W_{E,\alpha_4} & W_{E,\alpha_6} & \dots & W_{E,\alpha_n} \\ \psi_{\alpha_1} & W_{\alpha_1,\alpha_2} & W_{\alpha_1,\alpha_4} & W_{\alpha_1,\alpha_6} & \dots & W_{\alpha_1,\alpha_n} \\ \psi_{\alpha_3} & W_{\alpha_3,\alpha_2} & W_{\alpha_3,\alpha_4} & W_{\alpha_3,\alpha_6} & \dots & W_{\alpha_3,\alpha_n} \\ \psi_{\alpha_5} & W_{\alpha_5,\alpha_2} & W_{\alpha_5,\alpha_4} & W_{\alpha_5,\alpha_6} & \dots & W_{\alpha_5,\alpha_n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_{\alpha_{n-1}} & W_{\alpha_{n-1},\alpha_2} & W_{\alpha_{n-1},\alpha_4} & W_{\alpha_{n-1},\alpha_6} & \dots & W_{\alpha_{n-1},\alpha_n} \end{vmatrix}. \tag{11b}$$

The product $\prod_{i \neq j}^{m^2} (\alpha_i - \alpha_j)$ contains the same factors $(\alpha_i - \alpha_j)$ as those appearing in the definition of functions W_{α_i,α_j} (6).

If we put

$$\alpha_{i+1} - \alpha_i = \varepsilon_i \quad i = 1, 3, 5, \dots, n - 1$$

and impose the condition $\varepsilon_i \rightarrow 0$, we can verify the following limiting properties of the determinants $W_{2m}(x)$ and $W_{2m+1}(x)$:

$$\begin{aligned} \frac{W_{2m}(x)}{\varepsilon_1 \varepsilon_2 \dots \varepsilon_m} &\xrightarrow{\text{all } \varepsilon_i \rightarrow 0} \prod_{i \neq j}^m (\alpha_i - \alpha_j)^2 D_m(x) \\ \frac{W_{2m+1}(x)}{\varepsilon_1 \varepsilon_2 \dots \varepsilon_m} &\xrightarrow{\text{all } \varepsilon_i \rightarrow 0} - \prod_{i \neq j}^m (\alpha_i - \alpha_j)^2 \prod_{k=1}^m (E - \alpha_k) D_{m+1}(x) \end{aligned} \tag{12}$$

where $D_m(x)$ is the determinant of the symmetric matrix with elements $W_{\alpha_i,\alpha_j}(x)$. The determinant $D_{m+1}(x)$ is constructed from $D_m(x)$ by adding to its matrix a column of functions $\psi_E(x), \psi_{\alpha_1}(x), \psi_{\alpha_2}(x), \dots, \psi_{\alpha_m}(x)$ and a row of $W_{\alpha_i E}(x), W_{\alpha_2 E}(x), \dots, W_{\alpha_m E}(x)$. The potential difference $A(x)$ determined from formulae (5) becomes Bargman's formula [20]

and the function $\varphi_E(x)$ coincides with that given in the paper of Theis [21] (see also [4]). After decomposition of the last determinant on the first row we obtain

$$\varphi_E(x) = \psi_E(x) + D_m^{-1}(x) \sum_{i=1}^m W_{\alpha_i, E}(x) W^{(i, E)}(x) \tag{13}$$

where $W^{(i, E)}(x)$ is the co-factor of element $W_{\alpha_i, E}(x)$ in the determinant $D_{m+1}(x)$. According to equation (8) the last expression can be rewritten as formula (9) with $K(x, z)$ in the form

$$K(x, z) = \Psi^+(z)W(x) \tag{14}$$

where $\Psi^+(z)$ is the matrix row with elements $\psi_{\alpha_i}(z)$ and $W(x)$ is the matrix column with elements $W^{(i, E)}(x)D_m^{-1}(x)$.

According to the construction of the determinant $D_{m+1}(x)$, the co-factor $W^{(i, E)}(x)$ coincides with the determinant obtained from $D_m(x)$ by changing the i th column with that of functions $\psi_{\alpha_1}(x), \dots, \psi_{\alpha_m}(x)$. The decomposition of this determinant by the first column is consequently equal to the inverse matrix of the determinant $D_m(x)$ matrix multiplied by the column $\psi_{\alpha_1}(x), \dots, \psi_{\alpha_m}(x)$ with the sign changed. Thus, the kernel operator (14) is given by

$$K(x, z) = \Psi^+(z)D^{-1}(x)\Psi(x) \tag{15}$$

where $D(x)$ is the matrix of the determinant $D_m(x)$. Transformation (9) coincides in this case with that used in [1-4].

We have considered even $n = 2m$ in equation (5). In the case of odd $n = 2m + 1$, the limiting properties of $W_{2m+2}(x)$ and $W_{2m+1}(x)$ can easily be obtained from formula (11). The integral transformation in this case becomes

$$\varphi_E(x) = \int^x K(x, z)\psi_E(z) dz. \tag{16}$$

In the $n = 3$ case, the operator kernel $K(x, z)$ is written as

$$K(x, z) = \frac{W_{\alpha_1, \alpha_1}(x)\psi_{\alpha_3}(z) - W_{\alpha_1, \alpha_3}(x)\psi_{\alpha_1}(z)}{W_{\alpha_1, \alpha_1}(x)\psi_{\alpha_3}(x) - W_{\alpha_1, \alpha_3}(x)\psi_{\alpha_1}(x)}. \tag{17}$$

The potential difference $A(x)$ defined by the generalized Darboux transformation (see the second of equations (5)) is a regular function in (a, b) , if the Krein determinant $W_n(x)$ conserves its sign in interval (a, b) . When α_i are the points of the discrete spectrum of the regular Sturm-Liouville problem, the conservation sign condition of the determinant $W_n(x)$ is known [7]: $W_n(x) = W_*(\psi_{\alpha_{k_1}}, \dots, \psi_{\alpha_{k_n}})$ conserves its sign in (a, b) if the integers $(0 \leq)k_1 < k_2 < \dots < k_n$ satisfy the condition $(k - k_1)(k - k_2) \dots (k - k_n) \geq 0$ for all $k = 0, 1, 2, \dots$. This condition is true, in particular, if $\alpha_{k_1}, \dots, \alpha_{k_n}$ are two by two the juxtaposed points of the discrete spectrum. Values of α_i will be absent in the discrete spectrum of the new Hamiltonian H_1 . In the simplest case, we have $W_2(x) = W_*(\psi_n, \psi_{n+1}) \neq 0$ for all n . This property was recently rediscovered by Adler [18].

Another case when $W_2(x) \neq 0, x \in (a, b)$, corresponds to the limit $\alpha_2 \rightarrow \alpha_1$. The new potential $U(x) = V(x) + A(x)$ being determined from formulae (8) and (5) will depend on an arbitrary constant C but its discrete spectrum eigenvalues will not. When $C > 0$, the first relation (5) (or formula (9)) defines all discrete spectrum eigenfunctions of equation (2) if, for $\psi_E(x)$, those of equation (1) are taken. The discrete spectrum eigenvalues of H_1 will coincide with those of H_0 . If $C = 0$, the eigenstate of the Hamiltonian H_1 with energy $E = \alpha_1$ is excluded from its discrete spectrum. The solution of differential equation (2) with this value of E is $\varphi_E(x) = \psi_E(x)[\int^x \psi_E^2(z) dz]^{-1}$. This function does not obey the boundary condition of the discrete spectrum.

4. Examples

As an example, let us consider the first effective Coulomb potential

$$V(x) = -\frac{2z}{x} + \frac{l(l+1)}{x^2} \quad E_n = -\frac{z^2}{n^2} \quad n = 1, 2, 3, \dots \quad (18)$$

$$\psi_{nl}(x) = x^{l+1} \exp(-zx/n) L_{n-l-1}^{2l+1}(2zx/n)$$

where $L_n^\alpha(x)$ are generalized Laguerre polynomials.

The double Darboux transformation gives the simplest potential when the transformation functions have $l = 0$, $n_1 = 2$, $n_2 = 3$. The case $n_1 = 1$, $n_2 = 2$ corresponds to a double Darboux transformation with the ground state functions as transformation functions. It is well known [22] that the new potential in this case maintains the form of the initial one except for the value of l . By using formulae (5), we obtain a new potential

$$U(x) = -\frac{2z}{x} + \frac{10}{x^2} + \frac{100}{x^2}(3 - 2xz)Q_1^{-2}(xz) - \frac{40}{x^2}(2 - xz)Q_1^{-1}(xz) \quad (19)$$

$$Q_1(x) = 15 - 10x + 2x^2$$

and the wavefunctions of a new Hamiltonian. The ground state wavefunction is

$$\varphi_1(x) = N_1 x^3 \exp(-xz) Q_1^{-1}(xz).$$

The states with $n = 2$ and $n = 3$ are absent in the discrete spectrum of potential (19). When $n \geq 4$ we have the excited state functions

$$\varphi_{n-2}(x) = N_{n-2} x^3 \exp(-xz/n) P_{n-2}(xz) Q_1^{-1}(xz) \quad n = 4, 5, 6, \dots$$

N_i being the normalization constants and $P_n(x)$ the polynomials of order $n + 2$ whose connection with the Laguerre polynomials $L_n^\alpha(x)$ is established by formulae (5)

$$x^3 P_{n-2}(x) = 10n(-54 + 63x - 22x^2 + 2x^3) L_{n-2}^2(2x/n) \\ + L_{n-1}^1(2x/n)[270n(n-1) - 45(2 - 7n + 5n^2)x \\ + 10(6 - 11n + 5n^2)x^2 - 2(6 - 5n + n^2)x^3].$$

The first three polynomials are

$$P_1(x) = 1 \quad P_2(x) = 84 - 63x + 18x^2 - 2x^3 \\ P_3(x) = 875 - 700x + 220x^2 - 30x^3 + x^4.$$

The properties of $P_n(x)$ strongly differ from the properties of the Laguerre polynomials, though they are completely defined by the latter. For example, if we pass from $P_0(x)$ to $P_1(x)$, $P_2(x)$, ... the order of $P_n(x)$ varies from 0 to 3, 4, ... and there are no polynomials with orders equal to 1 and 2. But $P_n(x)$ has $n - 1$ zeros in the $(0, \infty)$ interval, and this fact corresponds well to the oscillator theorem for the Hamiltonian H_1 . A set of eigenfunctions of the new Hamiltonian obtained by the double Darboux transformation is complete in the space $L^2(R)$, ($R = [a, b]$) of square integrable on interval $[a, b]$ functions if the initial Hamiltonian has a complete set of eigenfunctions in the same space (in [18] this theorem is proved for an arbitrary initial potential).

The normalization constants N_i can easily be calculated due to the factorization property of the n -order Darboux transformation [11, 15]. We do not dwell on this calculation.

Consider now a triple Darboux transformation defined by formulae (16) and (17) with $n_1 = n_2 = 2$, $n_3 = 1$. The new potential for this case is

$$U(x) = -\frac{2z}{x} + \frac{2}{x^2} + 2x^8 z^{10} Q_2^{-2}(xz) + 2x^3 z^5 (4 - xz) Q_2^{-1}(xz) \quad (20)$$

$$Q_2(x) = 24 + 24x + 12x^2 + 4x^3 + x^4 + C \exp(x)$$

with an arbitrary parameter C . The form of potentials (20) is similar to the form of the Coulomb potential with $l \neq 0$, but in contrast to the latter there are two minima on the potential curve, and it tends to zero with increasing x more slowly. The first minimum is due to the attractive Coulomb part $-z/x$ and the second one to the presence of $\exp(x)$ in $Q_2(x)$. At $C \rightarrow 0$, this minimum moves to the left and merges with the first one. It is interesting to note that we actually have a family of isospectral potentials whose graphs, plotted for various values of constant C , resemble the diagram of a moving soliton. From formulae (5) we obtain the eigenfunctions of H_1 . The normalized-at-unity ground state function is written as

$$\varphi_2(x) = z^{5/2} \sqrt{C(C/4! + 1)} u^{-1}(x) \quad c > 0, c < -4!$$

For the normalized excited state functions we obtain

$$\varphi_n(x) = 2N_{n2}^0 n/z(n^2 - 4)^{-1/2} \exp(-xz/n) [x^2(3 - xz/n + \frac{1}{2}Q_2^{-1}(xz)R(xz)) \\ \times L_{n-3}^5(2xz/n) - (2xz^3/n)L_{n-4}^6(2xz/n)] \quad n = 3, 4, 5, \dots$$

$$R(x) = 96 + 72x + 24x^2 + 4x^3 + x^5 + C(4 - x)\exp(x)$$

where N_{n2}^0 is the normalization constant for the hydrogen-like function (18) at $l = 2$.

The level $n = 1$ is out of the discrete spectrum of Hamiltonian H_1 . When $C = 0$, the levels $n = 1$ and $n = 2$ are omitted. The function $\varphi_3(x)$ in the last case is the ground state function of potential (20).

Another simple example of a multiple Darboux transformation gives the harmonic oscillator with

$$V(x) = \frac{1}{4}x^2 - \frac{1}{2}.$$

Depending on the multiplicity of the transformation with the functions of the discrete spectrum

$$\psi_n(x) = \exp(-\frac{1}{4}x^2) H_n(x/\sqrt{2}) \quad n = 0, 1, 2, \dots$$

$H_n(x)$ being the Hermite polynomials, we can obtain the multiple-well potentials. For instance, the quadruple transformation with $n = 1, 2, 5, 6$ gives the triple-well potential

$$U(x) = \frac{7}{2} + \frac{1}{4}x^2 - 8Q_3^{-1}(x)(263 - 4x^2 - x^4 - 2x^6) \\ + 64Q_3^{-2}(x)(465 + 1035x^2 - 45x^4 + 17x^6)$$

$$Q_3(x) = 15 + 30x^2 + 2x^6 + x^8$$

with the first minimum at $x = 0$ with $U(0) = -4.5$ and two symmetric minima at $|x_{\min}| \approx 1.39$ with $U(x_{\min}) \approx 0.34$. There are two symmetric maxima at $|x_{\max}| \approx 0.76$ with $U(x_{\max}) \approx 2.72$. In the discrete spectrum of this potential the levels with $n = 1, 2, 5, 6$ are skipped. We do not cite the wavefunctions for this potential. It is not difficult to obtain them from equations (5).

Another interesting potential occurs for the single Darboux transformation with the function $\psi_{-1/2}(x) = {}_0F_1(\frac{3}{4}, \frac{1}{64}x^4)$ (${}_0F_1$ is the usual symbol for the hypergeometric function ${}_pF_q$), which is the solution of equation (1) for the harmonic oscillator potential with $E = -\frac{1}{2}$. This potential is a double well with $|x_{\min}| \approx 1.68$, $U(x_{\min}) \approx -0.94$ and with a maximum at $x = 0$ and $U(0) = -\frac{1}{2}$, i.e. the energetic level of the ground state touches the potential curve. In its discrete spectrum one additional level $E = -\frac{1}{2}$ occurs related to the spectrum of the harmonic oscillator potential with the ground state eigenfunction (up to a normalization factor) $\varphi_0(x) = \psi_{-1/2}^{-1}(x)$.

By choosing integral transformations (9), (15) or (16), (17) we can obtain asymmetric potentials with an oscillator spectrum. The simplest of them contains the probability integral erf(x) (or erfc(x)) (see e.g. [10]).

We will now give one significant example of a potential with n discrete energy levels disposed in a desirable manner obtained by the n -order Darboux transformation. This potential is cited in [11] but the normalized discrete spectrum wavefunctions are not. It should be noted that this potential (but in another form) is well known in soliton theory (see, for example, [23]) as the n -soliton potential for the KdV equation. The properties of n -soliton solutions of the KdV equation were studied by Wadati and Toda [24]. The term 'n-soliton potential' was introduced by Its and Matveev [25].

Consider the free-particle Schrödinger equation, $V(x) \equiv 0$. Choose, as transformation functions, the following set of solutions of equation (1):

$$\psi_{2i-1}(x) = \cosh(a_{2i-1}x + b_{2i-1}) \quad \psi_{2i}(x) = \sinh(a_{2i}x + b_{2i}) \quad i = 1, 2, \dots \quad (21)$$

where $a_{i+1} > a_i > 0$ and b_i are arbitrary real parameters. The number of transformation functions n can be both even and odd. It is remarkable that the Kreinian of this set, using elementary algebra, can be presented as a sum of hyperbolic cosines

$$W_*(\psi_1, \psi_2, \dots, \psi_n) = 2^{1-n} \sum_{k=1}^{2^{n-1}} B_k \cosh(\gamma_k x + \beta_k). \quad (22)$$

All coefficients γ_k have the form $\gamma_k = \sum_{i=1}^n c_i$, $c_i = \varepsilon a_i$, $\varepsilon = \pm 1$. The items in the sum (22) can be subdivided into groups. One has $n/2 + 1$ groups when n is even and $(n + 1)/2$ groups when n is odd. Every group except the last includes $\binom{n}{k}$ members; $k = 0, 1, \dots, n/2 - 1$ when n is even and $k = 0, 1, \dots, (n - 3)/2$ when n is odd. The last group contains $\frac{1}{2} \binom{n}{n/2}$ members when n is even and $\binom{n}{(n-1)/2}$ members when n is odd. Each group differs from another by the number of values of $\varepsilon = -1$. In the first group (comprised of only one item) all $\varepsilon = 1$, in the second group (n items) only one $\varepsilon = -1$ in each item and in the third group two $\varepsilon = -1$ in each item, etc. The coefficients β_k are constructed in the same manner but $\beta_k = \sum_{i=1}^n d_i$, $d_i = \varepsilon b_i$, $\varepsilon = \pm 1$. For the coefficients B_k we have: $B_k = \prod_{i>j}^{n^2} |c_i - c_j|$. It is interesting to note that after the absolute values of multipliers $(a_i - a_j)$ in B_k for $a_{i+1} > a_i$ are calculated, formula (22) becomes correct for arbitrary disposed a_i .

To calculate a new potential $A^{(n)}(x)$, it is sufficient now to take the second logarithmic derivative from the Kreinian (22) (see formula (5)).

The normalized-at-unity discrete spectrum wavefunctions are written as follows:

$$\varphi_i(x) = \left(\frac{1}{2} a_i \prod_{j=1(j \neq i)}^n |a_i^2 - a_j^2| \right)^{1/2} W_*^{(i)}(\psi_1, \dots, \psi_n) W_*^{-1}(\psi_1, \dots, \psi_n)$$

where $W_*^{(i)}(\psi_1, \dots, \psi_n)$ is the $(n - 1)$ -order Krein determinant constructed from the functions $\psi_1(x), \dots, \psi_n(x)$ except for the function $\psi_i(x)$. These functions have the property $H^{(n)}\varphi_i(x) = -a_i^2\varphi_i(x)$ so that they are enumerated in inverse order, $i = n$ corresponds to the ground state function.

We note that the potential thus obtained is reflectionless and coincides with the well known n -soliton potential. This fact can be established due to the following property: any factor $\exp(\gamma x)$ ($\gamma = \text{constant}$) for the Kreinian (20) does not affect the potential $A^{(n)}(x)$. In the case $n = 2$ we can easily obtain well known expressions for the two-soliton potential (see, for example, [23]).

The explicit expression for the three-level potential is:

$$\begin{aligned}
 V^{(3)}(x) = 8W^{-2}(x) & \left[- \sum_{i < j} a_i^2 a_j^2 (a_i^2 - a_j^2) \right. \\
 & + \sum_i (-1)^i (a_i^2 - a_k^2)(a_i^2 - a_j^2)(a_j^2 - a_k^2)^2 \cosh 2\theta_i \\
 & + \frac{1}{2} \sum_{i < j} (-1)^{i+j} a_k^2 (a_i^2 - a_k^2)(a_j^2 - a_k^2) \\
 & \left. \times [(a_i - a_j)^2 \cosh(2\theta_i + 2\theta_j) + (a_i + a_j)^2 \cosh(2\theta_i - 2\theta_j)] \right]
 \end{aligned}$$

where $\theta_i = a_i x + b_i$, $i, j, k = 1, 2, 3$, i, j, k are not equal to each other, the function $W(x)$ differs from the third-order Kreinian by the factor 4:

$$\begin{aligned}
 W(x) = 4W_*(\psi_1, \psi_2, \psi_3) = & (a_3 - a_1)(a_3 - a_2)(a_2 - a_1) \cosh(\theta_1 + \theta_2 + \theta_3) \\
 & + (a_3 + a_1)(a_3 - a_2)(a_2 + a_1) \cosh(-\theta_1 + \theta_2 + \theta_3) \\
 & + (a_3 - a_1)(a_3 + a_2)(a_2 + a_1) \cosh(\theta_1 - \theta_2 + \theta_3) \\
 & + (a_3 + a_1)(a_3 + a_2)(a_2 - a_1) \cosh(\theta_1 + \theta_2 - \theta_3).
 \end{aligned}$$

We will make some comments on the calculation of the normalization constants for the discrete spectrum eigenfunctions of isospectral Hamiltonians such as the n -soliton one. In the case where the absolute value of the transformation function denoted by $u(x)$ tends to infinity on both bounds of the interval $[a, b]$, the ground state eigenfunction of the new Hamiltonian after a single Darboux transformation is [4] (up to a normalization factor) $\varphi_0(x) = u^{-1}(x)$. It should be noted that the initial potential $V(x)$ can have no discrete spectrum at all (as in the case of the one-soliton potential). We then obtain a new potential with a single discrete spectrum level $E = E_0$. Two linearly independent solutions $u_1(x)$ and $u_2(x)$, from which the transformation function is composed, $u(x) = u_1(x) + C u_2(x)$, can always be chosen such that their Wronskian is equal to unity, $W(u_1, u_2) = 1$. This property is very useful for the calculation of the normalization constant for the function $\varphi_0(x)$ since it implies for the primitive of the function $\varphi_0^2(x)$ the following expression:

$$\int \varphi_0^2(x) dx = [C + u_1(x)/u_2(x)]^{-1}.$$

5. Conclusion

In conclusion it is necessary to note that the main result of this paper is the proof of the equivalence of the integral and differential exact solution stationary Schrödinger equation generation methods. This means that the integrals involved in these methods are calculable, and the use of either the differential or integral method is at the reader's discretion. Nevertheless it seems that the differential methods are preferable in quantum mechanics because with their help it is easier to obtain the normalized set of eigenfunctions.

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